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Sampling

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2.1 Introduction

To transmit analog message signals, such as speech signals or video signals, by digital means, the signal has to be converted into digital form. This process is known as analog-to-digital conversion. The sampling process is the first process performed in this conversion, and it converts a continuous-time signal into a discrete-time signal or a sequence of numbers. Digital transmission of analog signals is possible by virtue of the sampling theorem, and the sampling operation is performed in accordance with the sampling theorem.

In this chapter, using the Fourier transform technique, we present this remarkable sampling theorem and discuss the operation of sampling and practical aspects of sampling.

2.2 Instantaneous Sampling

Suppose we sample an arbitrary analog signal $m(t)$ shown in Fig. 2.1(a) instantaneously at a uniform rate, once every T_s seconds. As a result of this sampling process, we obtain an infinite sequence of samples $\{m(nT_s)\}$, where n takes on all possible integers. This form of sampling is called *instantaneous sampling*. We refer to T_s as the **sampling interval**, and its reciprocal $1/T_s = f_s$ as the **sampling rate.** Sampling rate (samples per second) is often cited in terms of sampling frequency expressed in hertz.

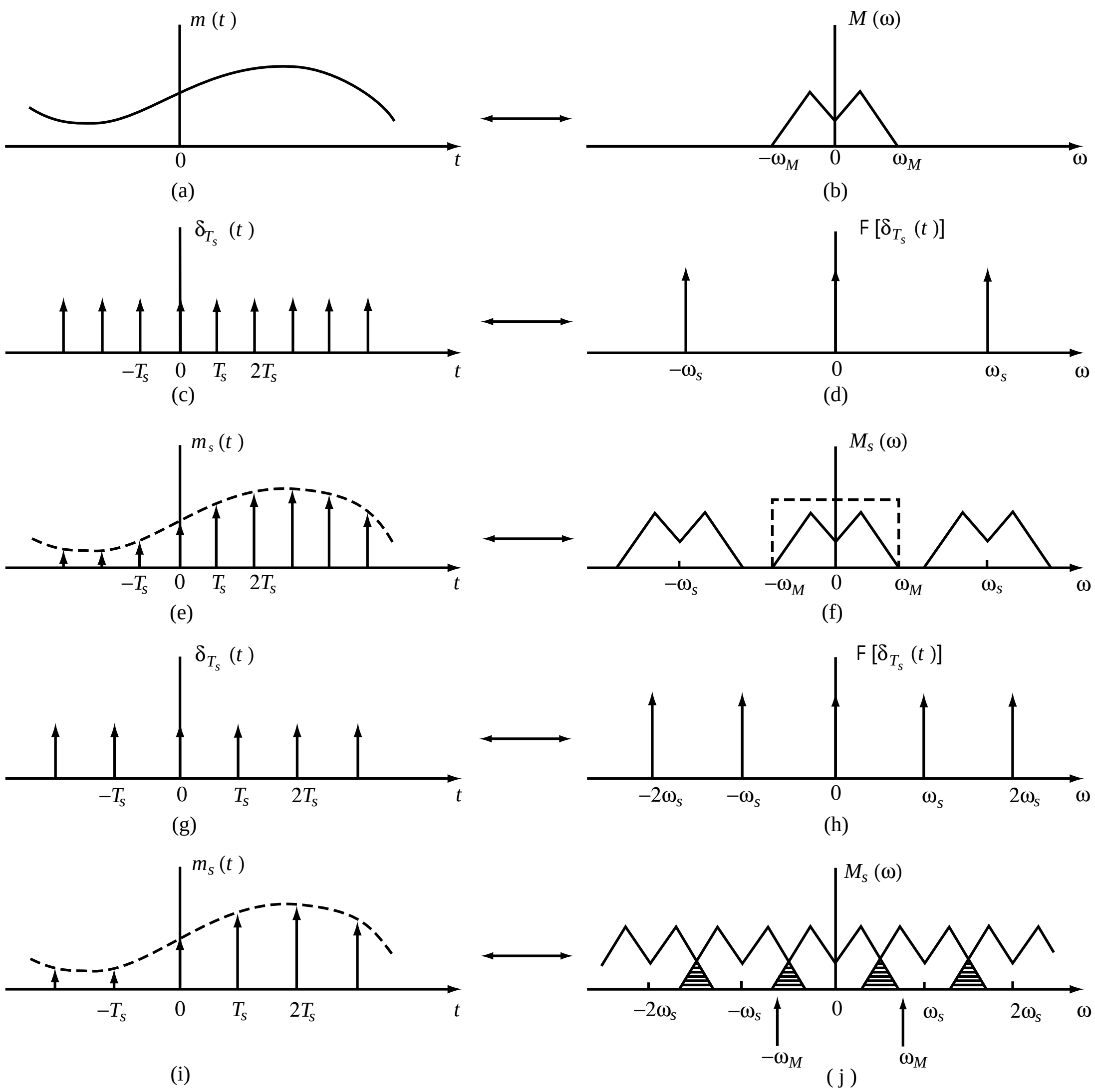

FIGURE 2.1: Illustration of instantaneous sampling and sampling theorem.

2.2.1 Ideal Sampled Signal

Let $m_s(t)$ be obtained by multiplication of $m(t)$ by the unit impulse train $\delta_T(t)$ with period T_s [Fig. 2.1(c)], that is,

$$\begin{aligned} m_s(t) &= m(t)\delta_{T_s}(t) = m(t)\sum_{n=-\infty}^{\infty}\delta\left(t-nT_s\right) \\ &= \sum_{n=-\infty}^{\infty} m(t)\delta\left(t-nT_s\right) = \sum_{n=-\infty}^{\infty} m\left(nT_s\right)\delta\left(t-nT_s\right) \end{aligned} \tag{2.1}$$

where we used the property of the δ function, $m(t)\delta(t-t_0) = m(t_0)\delta(t-t_0)$. The signal $m_s(t)$ [Fig. 2.1(e)] is referred to as the **ideal sampled signal.**

2.2.2 Band-Limited Signals

A real-valued signal $m(t)$ is called a **band-limited signal** if its Fourier transform $M(\omega)$ satisfies the condition

$$M(\omega) = 0 \qquad \text{for } |\omega| > \omega_M \tag{2.2}$$

where $\omega_M = 2\pi f_M$ [Fig. 2.1(b)]. A band-limited signal specified by Eq. (2.2) is often referred to as a *low-pass signal.*

2.3 Sampling Theorem

The sampling theorem states that a band-limited signal $m(t)$ specified by Eq. (2.2) can be uniquely determined from its values $m(nT_s)$ sampled at uniform interval T_s if $T_s \le \pi/\omega_M = 1/(2f_M)$. In fact, when $T_s = \pi/\omega_M$, $m(t)$ is given by

$$m(t) = \sum_{n=-\infty}^{\infty} m\left(nT_s\right) \frac{\sin \omega_M \left(t - nT_s\right)}{\omega_M \left(t - nT_s\right)} \tag{2.3}$$

which is known as the **Nyquist–Shannon interpolation formula** and it is also sometimes called the *cardinal series.* The sampling interval $T_s = 1/(2f_M)$is called the *Nyquist interval* and the minimum rate $f_s = 1/T_s = 2f_M$ is known as the **Nyquist rate.**

Illustration of the instantaneous sampling process and the sampling theorem is shown in Fig. 2.1. The Fourier transform of the unit impulse train is given by [Fig. 2.1(d)]

$$\mathcal{F}\left\{\delta_{T_s}(t)\right\} = \omega_s \sum_{n=-\infty}^{\infty} \delta\left(\omega - n\omega_s\right) \qquad \omega_s = 2\pi/T_s \tag{2.4}$$

Then, by the convolution property of the Fourier transform, the Fourier transform $M_s(\omega)$ of the ideal sampled signal $m_s(t)$ is given by

$$\begin{aligned} M_s(\omega) &= \frac{1}{2\pi}\left[M(\omega) * \omega_s \sum_{n=-\infty}^{\infty} \delta\left(\omega - n\omega_s\right)\right] \\ &= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} M\left(\omega - n\omega_s\right) \end{aligned} \tag{2.5}$$

where $*$ denotes convolution and we used the convolution property of the δ-function $M(\omega) * \delta(\omega - \omega_0) = M(\omega - \omega_0)$. Thus, the sampling has produced images of $M(\omega)$ along the frequency axis. Note that $M_s(\omega)$ will repeat periodically without overlap as long as $\omega_s \ge 2\omega_M$ or $f_s \ge 2f_M$ [Fig. 2.1(f)]. It is clear from Fig. 2.1(f) that we can recover $M(\omega)$ and, hence, $m(t)$ by passing the sampled signal $m_s(t)$ through an ideal low-pass filter having frequency response

$$H(\omega) = \begin{cases} T_s, & |\omega| \le \omega_M \\ 0, & \text{otherwise} \end{cases} \tag{2.6}$$

where $\omega_M = \pi / T_s$. Then

$$M(\omega) = M_s(\omega) H(\omega) \tag{2.7}$$

Taking the inverse Fourier transform of Eq. (2.6), we obtain the impulse response $h(t)$ of the ideal low-pass filter as

$$h(t) = \frac{\sin \omega_M t}{\omega_M t} \tag{2.8}$$

Taking the inverse Fourier transform of Eq. (2.7), we obtain

$$\begin{aligned} m(t) &= m_s(t) * h(t) \\ &= \sum_{n=-\infty}^{\infty} m(nT_s)\, \delta(t - nT_s) * \frac{\sin \omega_M t}{\omega_M t} \\ &= \sum_{n=-\infty}^{\infty} m(nT_s) \frac{\sin \omega_M (t - nT_s)}{\omega_M (t - nT_s)} \end{aligned} \tag{2.9}$$

which is Eq. (2.3).

The situation shown in Fig. 2.1(j) corresponds to the case where $f_s < 2f_M$. In this case there is an overlap between $M(\omega)$ and $M(\omega - \omega_M)$. This overlap of the spectra is known as *aliasing* or *foldover.* When this aliasing occurs, the signal is distorted and it is impossible to recover the original signal $m(t)$ from the sampled signal. To avoid aliasing, in practice, the signal is sampled at a rate slightly higher than the Nyquist rate. If $f_s > 2f_M$, then as shown in Fig. 2.1(f), there is a gap between the upper limit ω_M of $M(\omega)$ and the lower limit $\omega_s - \omega_M$ of $M(\omega - \omega_s)$. This range from ω_M to $\omega_s - \omega_M$ is called a *guard band.* As an example, speech transmitted via telephone is generally limited to f_M = 3.3 kHz (by passing the sampled signal through a low-pass filter). The Nyquist rate is, thus, 6.6 kHz. For digital transmission, the speech is normally sampled at the rate f_s = 8 kHz. The guard band is then $f_s - 2f_M$ = 1.4 kHz. The use of a sampling rate higher than the Nyquist rate also has the desirable effect of making it somewhat easier to design the low-pass reconstruction filter so as to recover the original signal from the sampled signal.

2.4 Sampling of Sinusoidal Signals

A special case is the sampling of a sinusoidal signal having the frequency f_M. In this case we require that $f_s > 2f_M$ rather that $f_s \geq 2f_M$. To see that this condition is necessary, let $f_s = 2f_M$. Now, if an initial sample is taken at the instant the sinusoidal signal is zero, then all successive samples will also be zero. This situation is avoided by requiring $f_s > 2f_M$.

2.5 Sampling of Bandpass Signals

A real-valued signal $m(t)$ is called a **bandpass signal** if its Fourier transform $M(\omega)$ satisfies the condition

$$M(\omega) = 0 \quad \text{except for} \quad \begin{cases} \omega_1 < \omega < \omega_2 \\ -\omega_2 < \omega < -\omega_1 \end{cases} \tag{2.10}$$

where $\omega_1 = 2\pi f_1$ and $\omega_2 = 2\pi f_2$ [Fig. 2.2(a)].

The sampling theorem for a band-limited signal has shown that a sampling rate of $2f_2$ or greater is adequate for a low-pass signal having the highest frequency f_2. Therefore, treating $m(t)$ specified by Eq. (2.10) as a special case of such a low-pass signal, we conclude that a sampling rate of $2f_2$ is

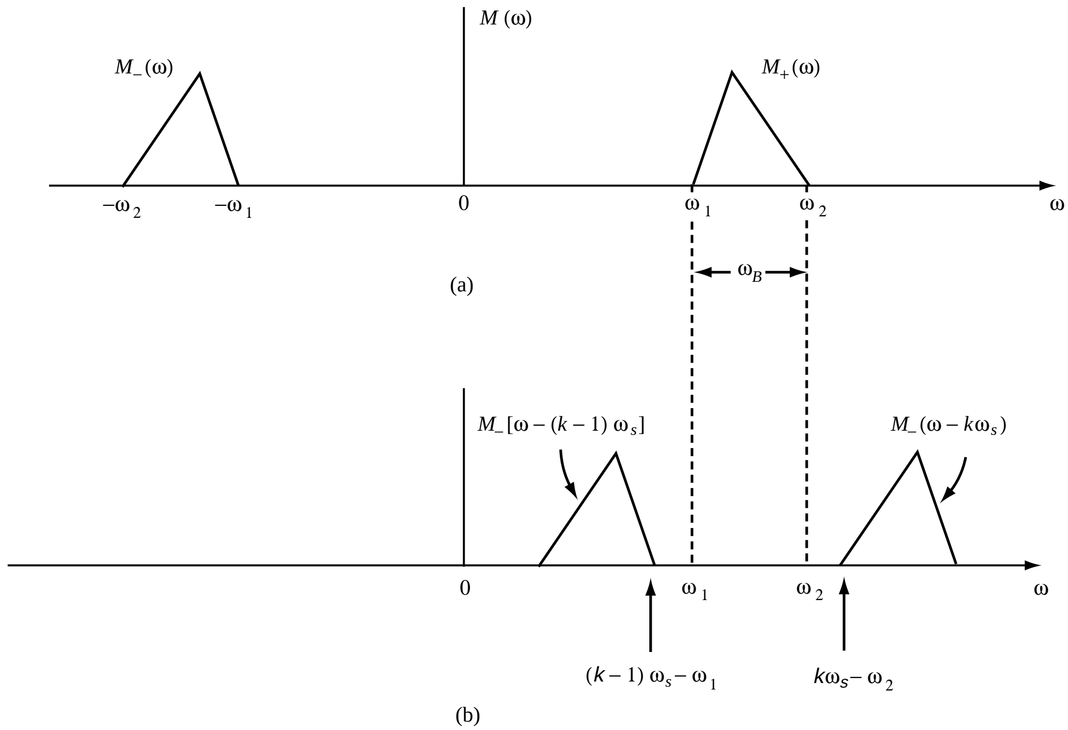

FIGURE 2.2: (a) Spectrum of a bandpass signal; (b) Shifted spectra of $M_{-}(\omega)$.

adequate for the sampling of the bandpass signal $m(t)$. But it is not necessary to sample this fast. The minimum allowable sampling rate depends on f_1, f_2, and the bandwidth $f_B = f_2 - f_1$.

Let us consider the direct sampling of the bandpass signal specified by Eq. (2.10). The spectrum of the sampled signal is periodic with the period $\omega_s = 2\pi f_s$, where f_s is the sampling frequency, as in Eq. (2.4). Shown in Fig. 2.2(b) are the two right shifted spectra of the negative side spectrum $M_{-}(\omega)$. If the recovering of the bandpass signal is achieved by passing the sampled signal through an ideal bandpass filter covering the frequency bands $(-\omega_2, -\omega_1)$ and (ω_1, ω_2), it is necessary that there be no aliasing problem. From Fig. 2.2(b), it is clear that to avoid overlap it is necessary that

$$\omega_s \geq 2(\omega_2 - \omega_1) \tag{2.11}$$

$$(k-1)\omega_s - \omega_1 \leq \omega_1 \tag{2.12}$$

and

$$k\omega_s - \omega_2 \geq \omega_2 \tag{2.13}$$

where $\omega_1 = 2\pi f_1$, $\omega_2 = 2\pi f_2$, and k is an integer ($k = 1, 2, \ldots$). Since $f_1 = f_2 - f_B$, these constraints can be expressed as

$$1 \leq k \leq \frac{f_2}{f_B} \leq \frac{k}{2}\frac{f_s}{f_B} \tag{2.14}$$

and

$$\frac{k-1}{2}\frac{f_s}{f_B} \leq \frac{f_2}{f_B} - 1 \tag{2.15}$$

A graphical description of Eqs. (2.14) and (2.15) is illustrated in Fig. 2.3. The unshaded regions represent where the constraints are satisfied, whereas the shaded regions represent the regions where the constraints are not satisfied and overlap will occur. The solid line in Fig. 2.3 shows the locus of the minimum sampling rate. The minimum sampling rate is given by

$$\min\{f_s\} = \frac{2f_2}{m} \tag{2.16}$$

where m is the largest integer not exceeding f_2/f_B. Note that if the ratio f_2/f_B is an integer, then the minimum sampling rate is $2f_B$. As an example, consider a bandpass signal with f_1 = 1.5 kHz and f_2 = 2.5 kHz. Here $f_B = f_2 - f_1 = 1$ kHz, and f_2/f_B = 2.5. Then from Eq. (2.16) and Fig. 2.3 we see that the minimum sampling rate is $2f_2/2 = f_2$ = 2.5 kHz, and allowable ranges of sampling rate are 2.5 kHz $\leq f_s \leq$ 3 kHz and $f_s \geq$ 5 kHz (= $2f_2$).

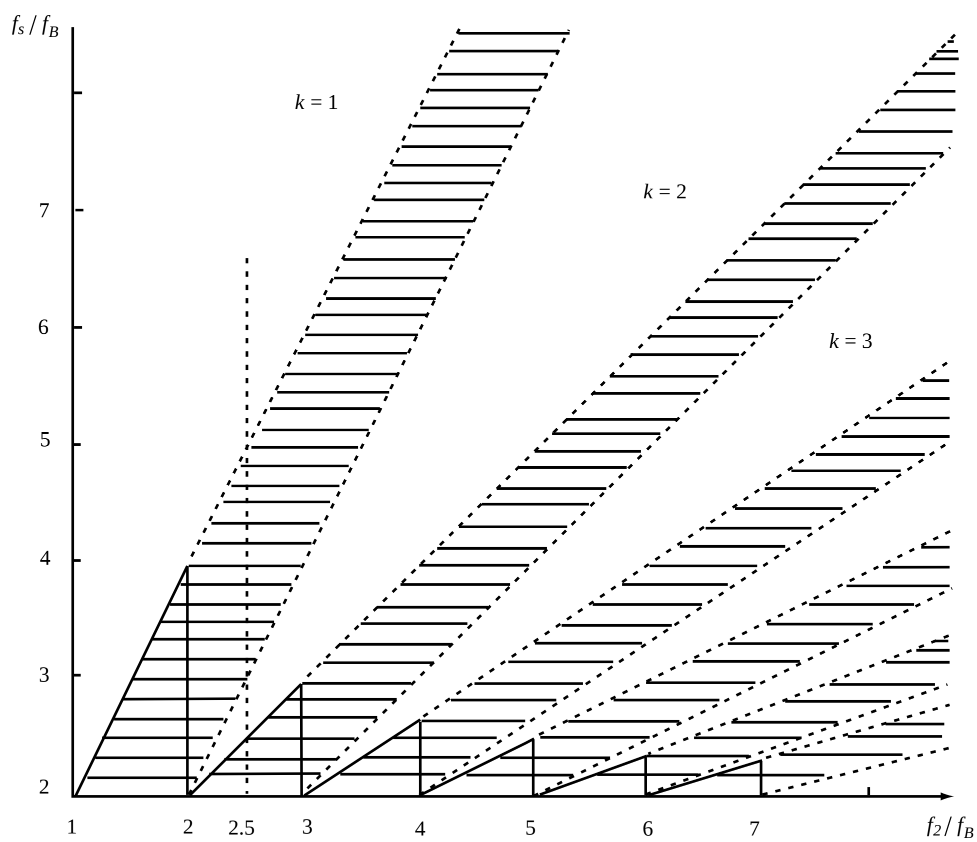

FIGURE 2.3: Minimum and permissible sampling rates for a bandpass signal.

2.6 Practical Sampling

In practice, the sampling of an analog signal is performed by means of high-speed switching circuits, and the sampling process takes the form of *natural sampling* or **flat-top sampling.**

2.6.1 Natural Sampling

Natural sampling of a band-limited signal $m(t)$ is shown in Fig. 2.4. The sampled signal $m_{\text{ns}}(t)$ can be expressed as

$$m_{\text{ns}}(t) = m(t)x_p(t) \tag{2.17}$$

where $x_p(t)$ is the periodic train of rectangular pulses with fundamental period T_s, and each rectangular pulse in $x_p(t)$ has duration d and unit amplitude [Fig. 2.4(b)]. Observe that the sampled signal $m_{\text{ns}}(t)$ consists of a sequence of pulses of varying amplitude whose tops follow the waveform of the signal $m(t)$ [Fig. 2.4(c)].

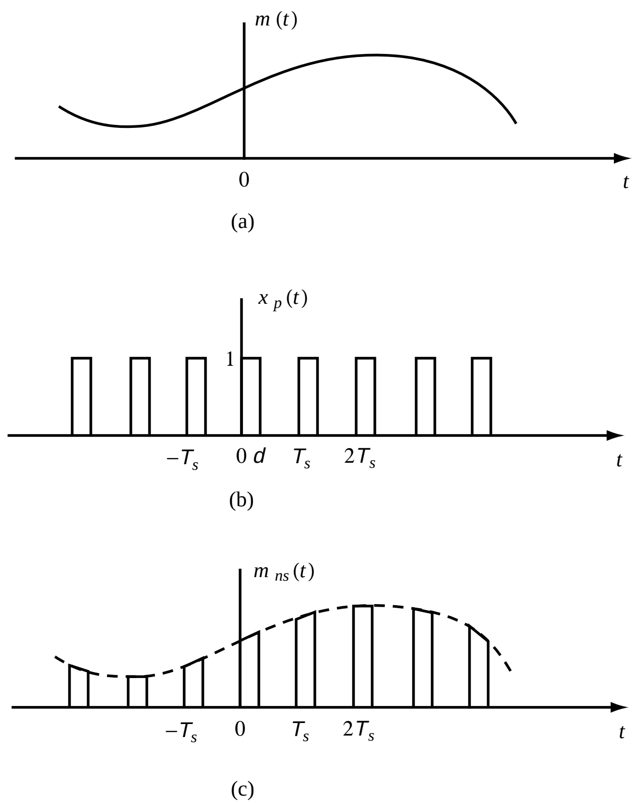

FIGURE 2.4: Natural sampling.

The Fourier transform of $x_p(t)$ is

$$X_p(\omega) = \sum_{n=-\infty}^{\infty} c_n \delta\left(\omega - n\omega_s\right) \qquad \omega_s = 2\pi / T_s \tag{2.18}$$

where

$$c_n = \frac{d}{T_s} \frac{\sin\left(n\omega_s d/2\right)}{n\omega_s d/2} e^{-jn\omega_s d/2} \tag{2.19}$$

Then the Fourier transform of $m_{\text{ns}}(t)$ is given by

$$M_{\text{ns}}(\omega) = M(\omega) * X_p(\omega) = \sum_{n=-\infty}^{\infty} c_n M\left(\omega - n\omega_s\right) \tag{2.20}$$

from which we see that the effect of the natural sampling is to multiply the nth shifted spectrum $M(\omega - n\omega_s)$ by a constant c_n. Thus, the original signal $m(t)$ can be reconstructed from $m_{\text{ns}}(t)$ with no distortion by passing $m_{\text{ns}}(t)$ through an ideal low-pass filter if the sampling rate f_s is equal to or greater than the Nyquist rate $2f_M$.

2.6.2 Flat-Top Sampling

The sampled waveform, produced by practical sampling devices that are the sample and hold types, has the form [Fig. 2.5(c)]

$$m_{\text{fs}}(t) = \sum_{n=-\infty}^{\infty} m\left(nT_s\right) p\left(t - nT_s\right) \tag{2.21}$$

where $p(t)$ is a rectangular pulse of duration d with unit amplitude [Fig. 2.5(a)]. This type of sampling is known as **flat-top sampling.** Using the ideal sampled signal $m_s(t)$ of Eq. (2.1), $m_{\text{fs}}(t)$ can be expressed as

$$m_{\text{fs}}(t) = p(t) * \left[\sum_{n=-\infty}^{\infty} m\left(nT_s\right) \delta\left(t - nT_s\right)\right] = p(t) * m_s(t) \tag{2.22}$$

Using the convolution property of the Fourier transform and Eq. (2.4), the Fourier transform of $m_{\text{fs}}(t)$ is given by

$$M_{\text{fs}}(\omega) = P(\omega) M_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} P(\omega) M\left(\omega - n\omega_s\right) \tag{2.23}$$

where

$$P(\omega) = d \frac{\sin\left(\omega d/2\right)}{\omega d/2} e^{-j\omega d/2} \tag{2.24}$$

From Eq.(2.23) we see that by using flat-top sampling we have introduced amplitude distortion and time delay, and the primary effect is an attenuation of high-frequency components. This effect is known as the *aperture effect*. The aperture effect can be compensated by an equalizing filter with a frequency response $H_{\text{eq}}(\omega) = 1/P(\omega)$. If the pulse duration d is chosen such that $d \ll T_s$, however, then $P(\omega)$ is essentially constant over the baseband and no equalization may be needed.

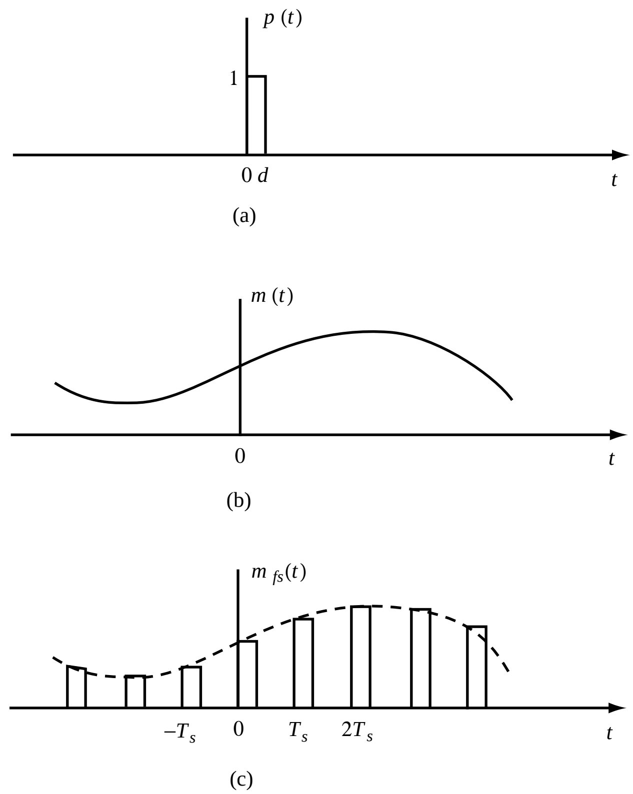

FIGURE 2.5: Flat-top sampling.

2.7 Sampling Theorem in the Frequency Domain

The sampling theorem expressed in Eq. (2.4) is the time-domain sampling theorem. There is a dual to this time-domain sampling theorem, i.e., the sampling theorem in the frequency domain.

Time-limited signals: A continuous-time signal $m(t)$ is called **time limited** if

$$m(t) = 0 \qquad \text{for} \quad |t| > |T_0| \tag{2.25}$$

Frequency-domain sampling theorem: The frequency-domain sampling theorem states that the Fourier transform $M(\omega)$ of a time-limited signal $m(t)$ specified by Eq. (2.25) can be uniquely determined from its values $M(n\omega_s)$ sampled at a uniform rate ω_s if $\omega_s \leq \pi/T_0$. In fact, when $\omega_s = \pi/T_0$, then $M(\omega)$ is given by

$$M(\omega) = \sum_{n=-\infty}^{\infty} M\left(n\omega_s\right) \frac{\sin T_0\left(\omega - n\omega_s\right)}{T_0\left(\omega - n\omega_s\right)} \tag{2.26}$$

2.8 Summary and Discussion

The sampling theorem is the fundamental principle of digital communications. We state the sampling theorem in two parts.

THEOREM 2.1 *If the signal contains no frequency higher than* f_M Hz, *it is completely described by specifying its samples taken at instants of time spaced* $1/2f_M$ s.

THEOREM 2.2 *The signal can be completely recovered from its samples taken at the rate of* $2f_M$ *samples per second or higher.*

The preceding sampling theorem assumes that the signal is strictly band limited. It is known that if a signal is band limited it cannot be time limited and vice versa. In many practical applications, the signal to be sampled is time limited and, consequently, it cannot be strictly band limited. Nevertheless, we know that the frequency components of physically occurring signals attenuate rapidly beyond some defined bandwidth, and for practical purposes we consider these signals are band limited. This approximation of real signals by band limited ones introduces no significant error in the application of the sampling theorem. When such a signal is sampled, we band limit the signal by filtering before sampling and sample at a rate slightly higher than the nominal Nyquist rate.

Defining Terms

Band-limited signal: A signal whose frequency content (Fourier transform) is equal to zero above some specified frequency.

Bandpass signal: A signal whose frequency content (Fourier transform) is nonzero only in a band of frequencies not including the origin.

Flat-top sampling: Sampling with finite width pulses that maintain a constant value for a time period less than or equal to the sampling interval. The constant value is the amplitude of the signal at the desired sampling instant.

Ideal sampled signal: A signal sampled using an ideal impulse train.

Nyquist rate: The minimum allowable sampling rate of $2f_M$ samples per second, to reconstruct a signal band limited to f_M hertz.

Nyquist-Shannon interpolation formula: The infinite series representing a time domain waveform in terms of its ideal samples taken at uniform intervals.

Sampling interval: The time between samples in uniform sampling.

Sampling rate: The number of samples taken per second (expressed in Hertz and equal to the reciprocal of the sampling interval).

Time-limited: A signal that is zero outside of some specified time interval.

References

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Further Information

For a tutorial review of the sampling theorem, historical notes, and earlier references see Jerri [6].