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Complex Envelope Representations for Modulated Signals[1]

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1.1 Introduction

What is a general representation for bandpass digital and analog signals? How do we represent a **modulated signal**? How do we evaluate the spectrum and the power of these signals? These are some of the questions that are answered in this chapter.

A *baseband* waveform has a spectral magnitude that is nonzero for frequencies in the vicinity of the origin (i.e., $f = 0$) and negligible elsewhere. A *bandpass* waveform has a spectral magnitude that is nonzero for frequencies in some band concentrated about a frequency $f = \pm f_c$ (where $f_c \gg 0$), and the spectral magnitude is negligible elsewhere. f_c is called the *carrier frequency*. The value of f_c may be arbitrarily assigned for mathematical convenience in some problems. In others, namely, **modulation** problems, f_c is the frequency of an oscillatory signal in the transmitter circuit and is the assigned frequency of the transmitter, such as 850 kHz for an AM broadcasting station.

In communication problems, the information source signal is usually a baseband signal—for example, a transistor-transistor logic (TTL) waveform from a digital circuit or an audio (analog) signal from a microphone. The communication engineer has the job of building a system that will transfer the information from this source signal to the desired destination. As shown in Fig. 1.1, this

[1]*Source:* Couch, Leon W., II. 1997. *Digital and Analog Communication Systems,* 5th ed., Prentice Hall, Upper Saddle River, NJ.

usually requires the use of a bandpass signal, $s(t)$, which has a bandpass spectrum that is concentrated at $\pm f_c$ where f_c is selected so that $s(t)$ will propagate across the communication channel (either a wire or a wireless channel).

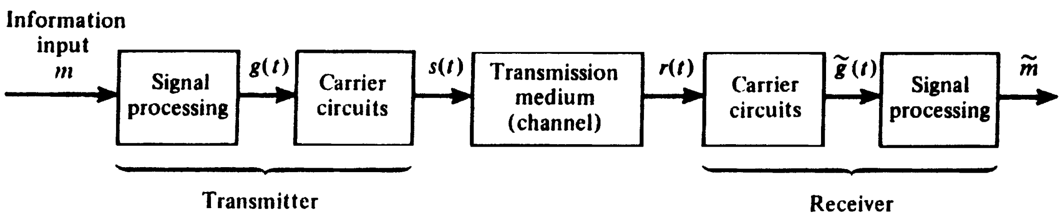

FIGURE 1.1: Bandpass communication system. *Source:* Couch, L.W., II. 1997. *Digital and Analog Communication Systems*, 5th ed., Prentice Hall, Upper Saddle River, NJ, p. 227. With permission.

Modulation is the process of imparting the source information onto a bandpass signal with a carrier frequency f_c by the introduction of amplitude and/or phase perturbations. This bandpass signal is called the *modulated* signal $s(t)$, and the baseband source signal is called the *modulating* signal $m(t)$. Examples of exactly how modulation is accomplished are given later in this chapter. This definition indicates that modulation may be visualized as a mapping operation that maps the source information onto the bandpass signal $s(t)$ that will be transmitted over the channel.

As the modulated signal passes through the channel, noise corrupts it. The result is a bandpass signal-plus-noise waveform that is available at the receiver input, $r(t)$, as illustrated in Fig. 1.1. The receiver has the job of trying to recover the information that was sent from the source; $\tilde{m}$ denotes the corrupted version of m.

1.2 Complex Envelope Representation

All bandpass waveforms, whether they arise from a modulated signal, interfering signals, or noise, may be represented in a convenient form given by the following theorem. $v(t)$ will be used to denote the **bandpass waveform** canonically. That is, $v(t)$ can represent the signal when $s(t) \equiv v(t)$, the noise when $n(t) \equiv v(t)$, the filtered signal plus noise at the channel output when $r(t) \equiv v(t)$, or any other type of bandpass waveform[2].

THEOREM 1.1 *Any physical bandpass waveform can be represented by*

$$v(t) = Re\left\{g(t)e^{j\omega_c t}\right\} \tag{1.1a}$$

$Re\{\cdot\}$ denotes the real part of $\{\cdot\}$. $g(t)$ is called the complex envelope of $v(t)$, and f_c is the associated carrier frequency (hertz) where $\omega_c = 2\pi f_c$. Furthermore, two other equivalent representations are

[2]The symbol $\equiv$ denotes an equivalence and the symbol $\triangleq$ denotes a definition.

$$v(t) = R(t) \cos [\omega_c t + \theta(t)] \tag{1.1b}$$

and

$$v(t) = x(t) \cos \omega_c t - y(t) \sin \omega_c t \tag{1.1c}$$

where

$$g(t) = x(t) + jy(t) = |g(t)|e^{j\angle g(t)} \equiv R(t)e^{j\theta(t)} \tag{1.2}$$

$$x(t) = Re\{g(t)\} \equiv R(t) \cos \theta(t) \tag{1.3a}$$

$$y(x) = Im\{g(t)\} \equiv R(t) \sin \theta(t) \tag{1.3b}$$

$$R(t) \triangleq |g(t)| \equiv \sqrt{x^2(t) + y^2(t)} \tag{1.4a}$$

$$\theta(t) \triangleq \angle g(t) = \tan^{-1} \left(\frac{y(t)}{x(t)} \right) \tag{1.4b}$$

The waveforms $g(t)$, $x(t)$, $y(t)$, $R(t)$, and $\theta(t)$ are all **baseband waveforms**, and, except for $g(t)$, they are all real waveforms. $R(t)$ is a nonnegative real waveform. Equation (1.1a–1.1c) is a low-pass-to-bandpass transformation. The $e^{j\omega_c t}$ factor in (1.1a) shifts (i.e., translates) the spectrum of the baseband signal $g(t)$ from baseband up to the carrier frequency f_c. In communications terminology the frequencies in the baseband signal $g(t)$ are said to be *heterodyned* up to f_c. The **complex envelope**, $g(t)$, is usually a complex function of time and it is the generalization of the phasor concept. That is, if $g(t)$ happens to be a complex constant, then $v(t)$ is a pure sine wave of frequency f_c and this complex constant is the phasor representing the sine wave. If $g(t)$ is not a constant, then $v(t)$ is not a pure sine wave because the amplitude and phase of $v(t)$ varies with time, caused by the variations of $g(t)$.

Representing the complex envelope in terms of two real functions in Cartesian coordinates, we have

$$g(x) \equiv x(t) + jy(t) \tag{1.5}$$

where $x(t) = \text{Re}\{g(t)\}$ and $y(t) = \text{Im}\{g(t)\}$. $x(t)$ is said to be the *in-phase modulation* associated with $v(t)$, and $y(t)$ is said to be the *quadrature modulation* associated with $v(t)$. Alternatively, the polar form of $g(t)$, represented by $R(t)$ and $\theta(t)$, is given by (1.2), where the identities between Cartesian and polar coordinates are given by (1.3a–1.3b) and (1.4a–1.4b). $R(t)$ and $\theta(t)$ are real waveforms and, in addition, $R(t)$ is always nonnegative. $R(t)$ is said to be the *amplitude modulation* (AM) on $v(t)$, and $\theta(t)$ is said to be the *phase modulation* (PM) on $v(t)$.

The usefulness of the complex envelope representation for bandpass waveforms cannot be overemphasized. In modern communication systems, the bandpass signal is often partitioned into two channels, one for $x(t)$ called the I (in-phase) channel and one for $y(t)$ called the Q (quadrature-phase) channel. In digital computer simulations of bandpass signals, the sampling rate used in the simulation can be minimized by working with the complex envelope, $g(t)$, instead of with the bandpass signal, $v(t)$, because $g(t)$ is the baseband equivalent of the bandpass signal [1].

1.3 Representation of Modulated Signals

Modulation is the process of encoding the source information $m(t)$ (modulating signal) into a bandpass signal $s(t)$ (modulated signal). Consequently, the modulated signal is just a special application of the bandpass representation. The *modulated signal* is given by

$$s(t) = \mathrm{Re}\left\{g(t)e^{j\omega_c t}\right\} \tag{1.6}$$

where $\omega_c = 2\pi f_c$. f_c is the carrier frequency. The complex envelope $g(t)$ is a function of the modulating signal $m(t)$. That is,

$$g(t) = g[m(t)] \tag{1.7}$$

Thus $g[\cdot]$ performs a mapping operation on $m(t)$. This was shown in Fig. 1.1.

Table 1.1 gives an overview of the *big picture* for the modulation problem. Examples of the mapping function $g[m]$ are given for amplitude modulation (AM), double-sideband suppressed carrier (DSB-SC), phase modulation (PM), frequency modulation (FM), single-sideband AM suppressed carrier (SSB-AM-SC), single-sideband PM (SSB-PM), single-sideband FM (SSB-FM), single-sideband envelope detectable (SSB-EV), single-sideband square-law detectable (SSB-SQ), and quadrature modulation (QM). For each $g[m]$, Table 1.1 also shows the corresponding $x(t)$ and $y(t)$ quadrature modulation components, and the corresponding $R(t)$ and $\theta(t)$ amplitude and phase modulation components. Digitally modulated bandpass signals are obtained when $m(t)$ is a digital baseband signal—for example, the output of a transistor transistor logic (TTL) circuit.

Obviously, it is possible to use other $g[m]$ functions that are not listed in Table 1.1. The question is: Are they useful? $g[m]$ functions are desired that are easy to implement and that will give desirable spectral properties. Furthermore, in the receiver the inverse function $m[g]$ is required. The inverse should be single valued over the range used and should be easily implemented. The inverse mapping should suppress as much noise as possible so that $m(t)$ can be recovered with little corruption.

1.4 Generalized Transmitters and Receivers

A more detailed description of transmitters and receivers as first shown in Fig. 1.1 will now be illustrated.

There are two canonical forms for the generalized transmitter, as indicated by (1.1b) and (1.1c). Equation (1.1b) describes an AM-PM type circuit as shown in Fig. 1.2. The baseband signal processing circuit generates $R(t)$ and $\theta(t)$ from $m(t)$. The R and θ are functions of the modulating signal $m(t)$, as given in Table 1.1, for the particular modulation type desired. The signal processing may be implemented either by using nonlinear analog circuits or a digital computer that incorporates the R and θ algorithms under software program control. In the implementation using a digital computer, one analog-to-digital converter (ADC) will be needed at the input of the baseband signal processor and two digital-to-analog converters (DACs) will be needed at the output. The remainder of the AM-PM canonical form requires radio frequency (RF) circuits, as indicated in the figure.

Figure 1.3 illustrates the second canonical form for the generalized transmitter. This uses in-phase and quadrature-phase (IQ) processing. Similarly, the formulas relating $x(t)$ and $y(t)$ to $m(t)$ are shown in Table 1.1, and the baseband signal processing may be implemented by using either analog hardware or digital hardware with software. The remainder of the canonical form uses RF circuits as indicated.

Analogous to the transmitter realizations, there are two canonical forms of receiver. Each one consists of RF carrier circuits followed by baseband signal processing as illustrated in Fig. 1.1. Typically

TABLE 1.1 Complex Envelope Functions for Various Types of Modulation[a]

Type of Modulation	Mapping Functions $g(m)$	Corresponding Quadrature Modulation $x(t)$	Corresponding Quadrature Modulation $y(t)$
AM	$A_c[1+m(t)]$	$A_c[1+m(t)]$	0
DSB-SC	$A_c m(t)$	$A_c m(t)$	0
PM	$A_c e^{jD_p m(t)}$	$A_c \cos[D_p m(t)]$	$A_c \sin[D_p m(t)]$
FM	$A_c e^{jD_f \int_{-\infty}^{t} m(\sigma)d\sigma}$	$A_c \cos\left[D_f \int_{-\infty}^{t} m(\sigma)d\sigma\right]$	$A_c \sin\left[D_f \int_{-\infty}^{t} m(\sigma)d\sigma\right]$
SSB-AM-SC[b]	$A_c[m(t) \pm j\hat{m}(t)]$	$A_c m(t)$	$\pm A_c \hat{m}(t)$
SSB-PM[b]	$A_c e^{jD_p[m(t) \pm j\hat{m}(t)]}$	$A_c e^{\mp D_p \hat{m}(t)} \cos[D_p m(t)]$	$A_c e^{\mp D_p \hat{m}(t)} \sin[D_p m(t)]$
SSB-FM[b]	$A_c e^{jD_f \int_{-\infty}^{t} [m(\sigma) \pm j\hat{m}(\sigma)]d\sigma}$	$A_c e^{\mp D_f \int_{-\infty}^{t} \hat{m}(\sigma)d\sigma} \cos\left[D_f \int_{-\infty}^{t} m(\sigma)d\sigma\right]$	$A_c e^{\mp D_f \int_{-\infty}^{t} \hat{m}(\sigma)d\sigma} \sin\left[D_f \int_{-\infty}^{t} m(\sigma)d\sigma\right]$
SSB-EV[b]	$A_c e^{\{\ln[1+m(t)] \pm j\hat{\ln}\vert 1+m(t)\vert\}}$	$A_c[1+m(t)] \cos\{\hat{\ln}[1+m(t)]\}$	$\pm A_c[1+m(t)] \sin\{\hat{\ln}[1+m(t)]\}$
SSB-SQ[b]	$A_c e^{(1/2)\{\ln[1+m(t)] \pm j\hat{\ln}\vert 1+m(t)\vert\}}$	$A_c \sqrt{1+m(t)} \cos\{\frac{1}{2}\hat{\ln}[1+m(t)]\}$	$\pm A_c \sqrt{1+m(t)} \sin\{\frac{1}{2}\hat{\ln}[1+m(t)]\}$
QM	$A_c[m_1(t)+jm_2(t)]$	$A_c m_1(t)$	$A_c m_2(t)$

TABLE 1.1 Complex Envelope Functions for Various Types of Modulation[a] *(Continued)*

Type of Modulation	Corresponding Amplitude and Phase Modulation: $R(t)$	$\theta(t)$	Linearity	Remarks
AM	$A_c\|1+m(t)\|$	$\begin{cases} 0, & m(t) > -1 \\ 180^\circ, & m(t) < -1 \end{cases}$	L[c]	$m(t) > -1$ required for envelope detection
DSB-SC	$A_c\|m(t)\|$	$\begin{cases} 0, & m(t) > 0 \\ 180^\circ, & m(t) < 0 \end{cases}$	L	Coherent detection required
PM	A_c	$D_p m(t)$	NL	D_p is the phase deviation constant (rad/volt)
FM	A_c	$D_f \int_{-\infty}^{t} m(\sigma)d\sigma$	NL	D_f is the frequency deviation constant (rad/volt-sec)
SSB-AM-SC[b]	$A_c\sqrt{[m(t)]^2+[\hat{m}(t)]^2}$	$\tan^{-1}[\pm\hat{m}(t)/m(t)]$	L	Coherent detection required
SSB-PM[b]	$A_c e^{\pm D_p \hat{m}(t)}$	$D_p m(t)$	NL	
SSB-FM[b]	$A_c e^{\pm D_f \int_{-\infty}^{t} \hat{m}(\sigma)d\sigma}$	$D_f \int_{-\infty}^{t} m(\sigma)d\sigma$	NL	
SSB-EV[b]	$A_c\|1+m(t)\|$	$\pm\hat{\ln}[1+m(t)]$	NL	$m(t) > -1$ is required so that the $\ln(\cdot)$ will have a real value
SSB-SQ[b]	$A_c\sqrt{1+m(t)}$	$\pm\frac{1}{2}\hat{\ln}[1+m(t)]$	NL	$m(t) > -1$ is required so that the $\ln(\cdot)$ will have a real value
QM	$A_c\sqrt{m_1^2(t)+m_2^2(t)}$	$\tan^{-1}[m_2(t)/m_1(t)]$	L	Used in NTSC color television; requires coherent detection

Source: Couch, L.W., II, 1997, *Digital and Analog Communication Systems,* 5th ed., Prentice Hall, Upper Saddle River, NJ, pp. 231-232. With permission.

[a] $A_c > 0$ is a constant that sets the power level of the signal as evaluated by use of (1.11); L, linear; NL, nonlinear; and $[\hat{\cdot}]$ is the Hilbert transform (a -90° phase-shifted version of $[\cdot]$). For example, $\hat{m}(t) = m(t) * \frac{1}{\pi t} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{m(\lambda)}{t-\lambda} d\lambda$.

[b] Use upper signs for upper sideband signals and lower signals for lower sideband signals.

[c] In the strict sense, AM signals are not linear because the carrier term does not satisfy the linearity (superposition) condition.

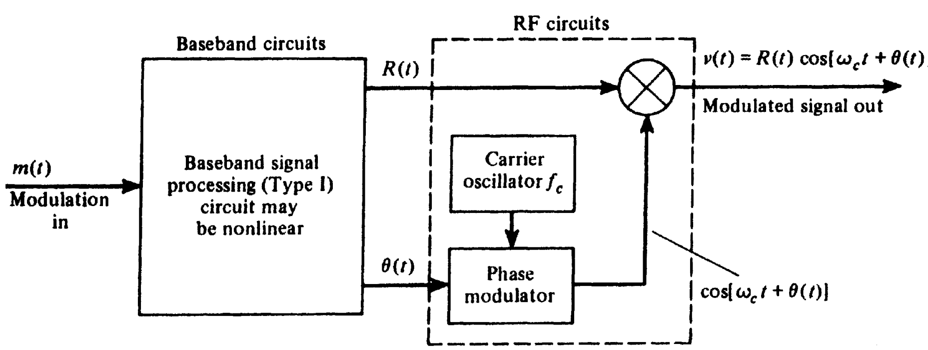

FIGURE 1.2: Generalized transmitter using the AM-PM generation technique. *Source:* Couch, L.W., II. 1997. *Digital and Analog Communication Systems,* 5th ed., Prentice Hall, Upper Saddle River, NJ, p. 278. With permission.

the carrier circuits are of the superheterodyne-receiver type which consist of an RF amplifier, a down converter (mixer plus local oscillator) to some intermediate frequency (IF), an IF amplifier and then detector circuits [1]. In the first canonical form of the receiver, the carrier circuits have amplitude and phase detectors that output $\tilde{R}(t)$ and $\tilde{\theta}(t)$, respectively. This pair, $\tilde{R}(t)$ and $\tilde{\theta}(t)$, describe the polar form of the received complex envelope, $\tilde{g}(t)$. $\tilde{R}(t)$ and $\tilde{\theta}(t)$ are then fed into the signal processor

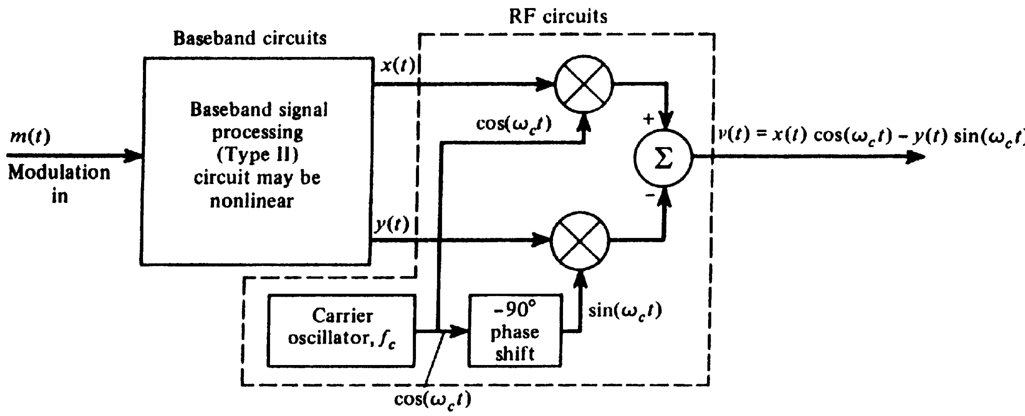

FIGURE 1.3: Generalized transmitter using the quadrature generation technique. *Source:* Couch, L.W., II. 1997. *Digital and Analog Communication Systems*, 5th ed., Prentice Hall, Upper Saddle River, NJ, p. 278. With permission.

which uses the inverse functions of Table 1.1 to generate the recovered modulation, $\tilde{m}(t)$. The second canonical form of the receiver uses quadrature product detectors in the carrier circuits to produce the Cartesian form of the received complex envelope, $\tilde{x}(t)$ and $\tilde{y}(t)$. $\tilde{x}(t)$ and $\tilde{y}(t)$ are then inputted to the signal processor which generates $\tilde{m}(t)$ at its output.

Once again, it is stressed that any type of signal modulation (see Table 1.1) may be generated (transmitted) or detected (received) by using either of these two canonical forms. Both of these forms conveniently separate baseband processing from RF processing. Digital techniques are especially useful to realize the baseband processing portion. Furthermore, if digital computing circuits are used, any desired modulation type can be realized by selecting the appropriate software algorithm.

1.5 Spectrum and Power of Bandpass Signals

The spectrum of the bandpass signal is the translation of the spectrum of its complex envelope. Taking the **Fourier transform** of (1.1a), the spectrum of the bandpass waveform is [1]

$$V(f) = \frac{1}{2}\left[G\left(f - f_c\right) + G^*\left(-f - f_c\right)\right] \tag{1.8}$$

where $G(f)$ is the Fourier transform of $g(t)$,

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt\ ,$$

and the asterisk superscript denotes the complex conjugate operation. The *power spectra density* (PSD) of the bandpass waveform is [1]

$$\mathcal{P}_v(f) = \frac{1}{4}\left[\mathcal{P}_g\left(f - f_c\right) + \mathcal{P}_g\left(-f - f_c\right)\right] \tag{1.9}$$

where $\mathcal{P}_g(f)$ is the PSD of $g(t)$.

The average power dissipated in a resistive load is $V_{\rm rms}^2/R_L$ or $I_{\rm rms}^2 R_L$ where $V_{\rm rms}$ is the rms value of the voltage waveform across the load and $I_{\rm rms}$ is the rms value of the current through the load. For bandpass waveforms, Equation (1.1a–1.1c) may represent either the voltage or the current. Furthermore, the rms values of $v(t)$ and $g(t)$ are related by [1]

$$v_{\mathrm{rms}}^2 = \left\langle v^2(t) \right\rangle = \frac{1}{2}\left\langle |g(t)|^2 \right\rangle = \frac{1}{2} g_{\mathrm{rms}}^2 \tag{1.10}$$

where $\langle \cdot \rangle$ denotes the time average and is given by

$$\left\langle \left[\quad \right] \right\rangle = \lim_{t \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[\quad \right] dt$$

Thus, if $v(t)$ of (1.1a–1.1c) represents the bandpass voltage waveform across a resistive load, the average power dissipated in the load is

$$P_L = \frac{v_{\mathrm{rms}}^2}{R_L} = \frac{\langle v^2(t) \rangle}{R_L} = \frac{\langle |g(t)|^2 \rangle}{2R_L} = \frac{g_{\mathrm{rms}}^2}{2R_L} \tag{1.11}$$

where g_{rms} is the rms value of the complex envelope and R_L is the resistance of the load.

1.6 Amplitude Modulation

Amplitude modulation (AM) will now be examined in more detail. From Table 1.1 the complex envelope of an AM signal is

$$g(t) = A_c[1 + m(t)] \tag{1.12}$$

so that the spectrum of the complex envelope is

$$G(f) = A_c\delta(f) + A_c M(f) \tag{1.13}$$

Using (1.6), we obtain the AM signal waveform

$$s(t) = A_c[1 + m(t)] \cos \omega_c t \tag{1.14}$$

and, using (1.8), the AM spectrum

$$S(f) = \frac{1}{2} A_c \left[\delta\left(f - f_c\right) + M\left(f - f_c\right) + \delta\left(f + f_c\right) + M\left(f + f_c\right)\right] \tag{1.15}$$

where $\delta(f) = \delta(-f)$ and, because $m(t)$ is real, $M^*(f) = M(-f)$. Suppose that the magnitude spectrum of the modulation happens to be a triangular function, as shown in Fig. 1.4(a). This spectrum might arise from an analog audio source where the bass frequencies are emphasized. The resulting AM spectrum, using (1.15), is shown in Fig. 1.4(b). Note that because $G(f - f_c)$ and $G^*(-f - f_c)$ do not overlap, the magnitude spectrum is

$$|S(f)| = \begin{cases} \frac{1}{2} A_c \delta\left(f - f_c\right) + \frac{1}{2} A_c \left|M\left(f - f_c\right)\right|, & f > 0 \\ \frac{1}{2} A_c \delta\left(f + f\right)_c + \frac{1}{2} A_c \left|M\left(-f - f\right)_c\right|, & f < 0 \end{cases} \tag{1.16}$$

The 1 in

$$g(t) = A_c[1 + m(t)]$$

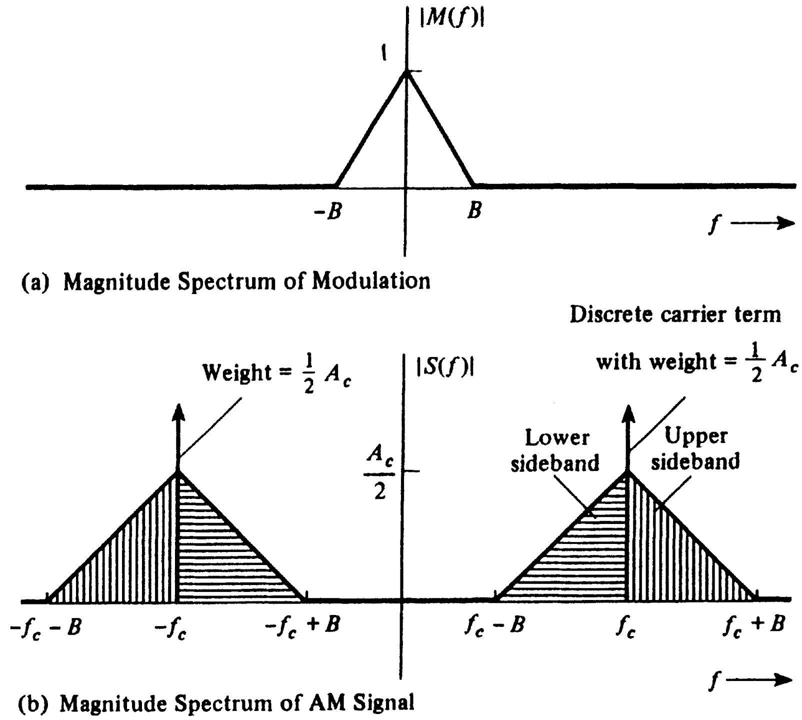

FIGURE 1.4: Spectrum of an AM signal. *Source:* Couch, L.W., II. 1997. *Digital and Analog Communication Systems,* 5th ed., Prentice Hall, Upper Saddle River, NJ, p. 235. With permission.

causes delta functions to occur in the spectrum at $f = \pm f_c$, where f_c is the assigned carrier frequency. Also, from Fig. 1.4 and (1.16), it is realized that the bandwidth of the AM signal is 2B. That is, the bandwidth of the AM signal is twice the bandwidth of the baseband modulating signal.

The average power dissipated into a resistive load is found by using (1.11).

$$P_L = \frac{A_c^2}{2R_L}\left\langle |1+m(t)|^2 \right\rangle = \frac{A_c^2}{2R_L}\left[1 + 2\langle m(t)\rangle + \left\langle m^2(t)\right\rangle\right]$$

If we assume that the dc value of the modulation is zero, $\langle m(t)\rangle = 0$, then the average power dissipated into the load is

$$P_L = \frac{A_c^2}{2R_L}\left\langle 1 + m_{\text{rms}}^2 \right\rangle \tag{1.17}$$

where m_{rms} is the rms value of the modulation, $m(t)$. Thus, the average power of an AM signal changes if the rms value of the modulating signal changes. For example, if $m(t)$ is a sine wave test tone with a peak value of 1.0 for 100% modulation,

$$m_{\text{rms}} = 1/\sqrt{2}\,.$$

Assume that $A_c = 1000$ volts and $R_L = 50$ ohms, which are typical values used in AM broadcasting. Then the averagepower dissipated into the 50 Ω load for this AM signal is

$$P_L = \frac{(1000)^2}{2(50)}\left[1 + \frac{1}{2}\right] = 15{,}000 \text{ watts} \tag{1.18}$$

The Federal Communications Commission (FCC) rated carrier power is obtained when $m(t) = 0$. In this case, (1.17) becomes $P_L = (1000)^2/100 = 10{,}000$ watts and the FCC would rate this as a 10,000 watt AM station. The sideband power for 100% sine wave modulation is 5,000 watts.

Now let the modulation on the AM signal be a binary digital signal such that $m(t) = \pm 1$ where $+1$ is used for a binary one and -1 is used for a binary 0. Referring to (1.14), this AM signal becomes an *on-off keyed* (OOK) digital signal where the signal is on when a binary one is transmitted and off when a binary zero is transmitted. For $A_c = 1000$ and $R_L = 50\ \Omega$, the average power dissipated would be 20,000 watts since $m_{\text{rms}} = 1$ for $m(t) = \pm 1$.

1.7 Phase and Frequency Modulation

Phase modulation (PM) and *frequency modulation* (FM) are special cases of angle-modulated signalling. In angle-modulated signalling the complex envelope is

$$g(t) = A_c e^{j\theta(t)} \tag{1.19}$$

Using (1.6), the resulting *angle-modulated* signal is

$$s(t) = A_c \cos\left[\omega_c + \theta(t)\right] \tag{1.20}$$

For PM the phase is directly proportional to the modulating signal:

$$\theta(t) = D_p m(t) \tag{1.21}$$

where the proportionality constant D_p is the phase sensitivity of the phase modulator, having units of radians per volt [assuming that $m(t)$ is a voltage waveform]. For FM the phase is proportional to the integral of $m(t)$:

$$\theta(t) = D_f \int_{-\infty}^{t} m(\sigma)d\sigma \tag{1.22}$$

where the frequency deviation constant D_f has units of radians/volt-second. These concepts are summarized by the PM and FM entries in Table 1.1.

By comparing the last two equations, it is seen that if we have a PM signal modulated by $m_p(t)$, there is *also* FM on the signal corresponding to a *different* modulating waveshape that is given by

$$m_f(t) = \frac{D_p}{D_f}\left[\frac{dm_p(t)}{dt}\right] \tag{1.23}$$

where the subscripts f and p denote frequency and phase, respectively. Similarly, if we have an FM signal modulated by $m_f(t)$, the corresponding phase modulation on this signal is

$$m_p(t) = \frac{D_f}{D_p}\int_{-\infty}^{t} m_f(\sigma)d\sigma \tag{1.24}$$

By using (1.24), a PM circuit may be used to synthesize an FM circuit by inserting an integrator in cascade with the phase modulator input.

Other properties of PM and FM are that the **real envelope**, $R(t) = |g(t)| = A_c$, is a constant, as seen from (1.19). Also, $g(t)$ is a *nonlinear* function of the modulation. However, from (1.21) and (1.22), $\theta(t)$ is a linear function of the modulation, $m(t)$. Using (1.11), the average power dissipated by a PM or FM signal is the constant

$$P_L = \frac{A_c^2}{2R_L} \tag{1.25}$$

That is, the average power of a PM or FM signal does not depend on the modulating waveform, $m(t)$.

The *instantaneous frequency deviation* for an FM signal from its carrier frequency is given by the derivative of its phase $\theta(t)$. Taking the derivative of (1.22), the *peak frequency deviation* is

$$\Delta F = \frac{1}{2\pi} D_f M_p \text{ Hz} \tag{1.26}$$

where $M_p = \max[m(t)]$ is the peak value of the modulation waveform and the derivative has been divided by 2π to convert from radians/sec to Hz units.

For FM and PM signals, Carson's rule estimates the transmission bandwidth containing approximately 98% of the total power. This FM or PM signal bandwidth is

$$B_T = 2(\beta + 1)B \tag{1.27}$$

where B is bandwidth (highest frequency) of the modulation. The modulation index β, is $\beta = \Delta F/B$ for FM and $\beta = \max[D_p m(t)] = D_p M_p$ for PM.

The AMPS (Advanced Mobile Phone System) analog cellular phones use FM signalling. A peak deviation of 12 kHz is specified with a modulation bandwidth of 3 kHz. From (1.27), this gives a bandwidth of 30 kHz for the AMPS signal and allows a channel spacing of 30 kHz to be used. To accommodate more users, narrow-band AMPS (NAMPS) with a 5 kHz peak deviation is used in some areas. This allows 10 kHz channel spacing if the carrier frequencies are carefully selected to minimize interference to used adjacent channels. A maximum FM signal power of 3 watts is allowed for the AMPS phones. However, hand-held AMPS phones usually produce no more than 600 mW which is equivalent to 5.5 volts rms across the 50 Ω antenna terminals.

The GSM (Group Special Mobile) digital cellular phones use FM with *minimum frequency-shift-keying* (MSK) where the peak frequency deviation is selected to produce orthogonal waveforms for binary one and binary zero data. (Digital phones use a speech codec to convert the analog voice source to a digital data source for transmission over the system.) Orthogonality occurs when $\Delta F = 1/4R$ where R is the bit rate (bits/sec) [1]. Actually, GSM uses Gaussian shaped MSK (GMSK). That is, the digital data waveform (with rectangular binary one and binary zero pulses) is first filtered by a low-pass filter having a Gaussian shaped frequency response (to attenuate the higher frequencies). This Gaussian filtered data waveform is then fed into the frequency modulator to generate the GMSK signal. This produces a digitally modulated FM signal with a relatively small bandwidth.

Other digital cellular standards use QPSK signalling as discussed in the next section.

1.8 QPSK Signalling

Quadrature phase-shift-keying (QPSK) is a special case of quadrature modulation as shown in Table 1.1 where $m_1(t) = \pm 1$ and $m_2(t) = \pm 1$ are two binary bit streams. The complex envelope for QPSK is

$$g(t) = x(t) + jy(t) = A_c[m_1(t) + jm_2(t)]$$

where $x(t) = \pm A_c$ and $y(t) = \pm A_c$. The permitted values for the complex envelope are illustrated by the QPSK **signal constellation** shown in Fig. 1.5a. The *signal constellation* is a plot of the permitted values for the complex envelope, $g(t)$. QPSK may be generated by using the quadrature generation technique of Fig. 1.3 where the baseband signal processor is a serial-to-parallel converter that reads in two bits of data at a time from the serial binary input stream, $m(t)$ and outputs the first of the two bits to $x(t)$ and the second bit to $y(t)$. If the two input bits are both binary ones, (11), then $m_1(t) = +A_c$ and $m_2(t) = +A_c$. This is represented by the top right-hand dot for $g(t)$ in the signal constellation for QPSK signalling in Fig. 1.5a. Likewise, the three other possible two-bit words, (10), (01), and (00), are also shown. The QPSK signal is also equivalent to a four-phase phase-shift-keyed signal (4PSK) since all the points in the signal constellation fall on a circle where the permitted phases are $\theta(t) = 45°,\ 135°,\ 225°$, and $315°$. There is no amplitude modulation on the QPSK signal since the distances from the origin to all the signal points on the signal constellation are equal.

For QPSK, the spectrum of $g(t)$ is of the $\sin x/x$ type since $x(t)$ and $y(t)$ consists of rectangular data pulses of value $\pm A_c$. Moreover, it can be shown that for equally likely independent binary one and binary zero data, the power spectral density of $g(t)$ for digitally modulated signals with M point signal constellations is [1]

$$\mathcal{P}_g(f) = K\left(\frac{\sin \pi f \ell T_b}{\pi f \ell T_b}\right)^2 \tag{1.28}$$

where K is a constant, $R = 1/T_b$ is the data rate (bits/sec) of $m(t)$ and $M = 2^\ell$. M is the number of points in the signal constellation. For QPSK, $M = 4$ and $\ell = 2$. This PSD for the complex envelope, $\mathcal{P}_g(f)$, is plotted in Fig. 1.6. The PSD for the QPSK signal ($\ell = 2$) is given by translating $\mathcal{P}_g(f)$ up to the carrier frequency as indicated by (1.9).

Referring to Fig. 1.6 or using (1.28), the first-null bandwidth of $g(t)$ is R/ℓ Hz. Consequently, the null-to-null bandwidth of the modulated RF signal is

$$B_{\text{null}} = \frac{2R}{\ell}\ \text{Hz} \tag{1.29}$$

For example, if the data rate of the baseband information source is 9600 bits/sec, then the null-to-null bandwidth of the QPSK signal would be 9.6 Hz since $\ell = 2$.

Referring to Fig. 1.6, it is seen that the sidelobes of the spectrum are relatively large so, in practice, the sidelobes of the spectrum are filtered off to prevent interference to the adjacent channels. This filtering rounds off the edges of the rectangular data pulses and this causes some amplitude modulation on the QPSK signal. That is, the points in the signal constellation for the filtered QPSK signal would be fuzzy since the transition from one constellation point to another point is not instantaneous because the filtered data pulses are not rectangular. QPSK is the modulation used for digital cellular phones with the IS-95 Code Division Multiple Access (CDMA) standard.

Equation (1.28) and Fig. 1.6 also represent the spectrum for *quadrature modulation amplitude modulation* (QAM) signalling. QAM signalling allows more than two values for $x(t)$ and $y(t)$. For example QAM where $M = 16$ has 16 points in the signal constellation with 4 values for $x(t)$ and 4 values for $y(t)$ such as, for example, $x(t) = +A_c,\ -A_c,\ +3A_c,\ -3A_c$ and $y(t) = +A_c,\ -A_c,\ +3A_c,\ -3A_c$. This is shown in Fig. 1.5b. Each point in the $M = 16$ QAM signal constellation would represent a unique four-bit data word, as compared with the $M = 4$ QPSK signal constellation shown in Fig. 1.5a where each point represents a unique two-bit data word. For a $R = 9600$ bits/sec information source data rate, a $M = 16$ QAM signal would have a null-to-null bandwidth of 4.8 kHz since $\ell = 4$.

For OOK signalling as described at the end of Section 1.6, the signal constellation would consist of $M = 2$ points along the x axis where $x = 0, 2A_c$ and $y = 0$. This is illustrated in Fig. 1.5c. For a $R = 9600$ bit/sec information source data rate, an OOK signal would have a null-to-null bandwidth of 19.2 kHz since $\ell = 1$.

Defining Terms

Bandpass waveform: The spectrum of the waveform is nonzero for frequencies in some band concentrated about a frequency $f_c \gg 0$; f_c is called the carrier frequency.

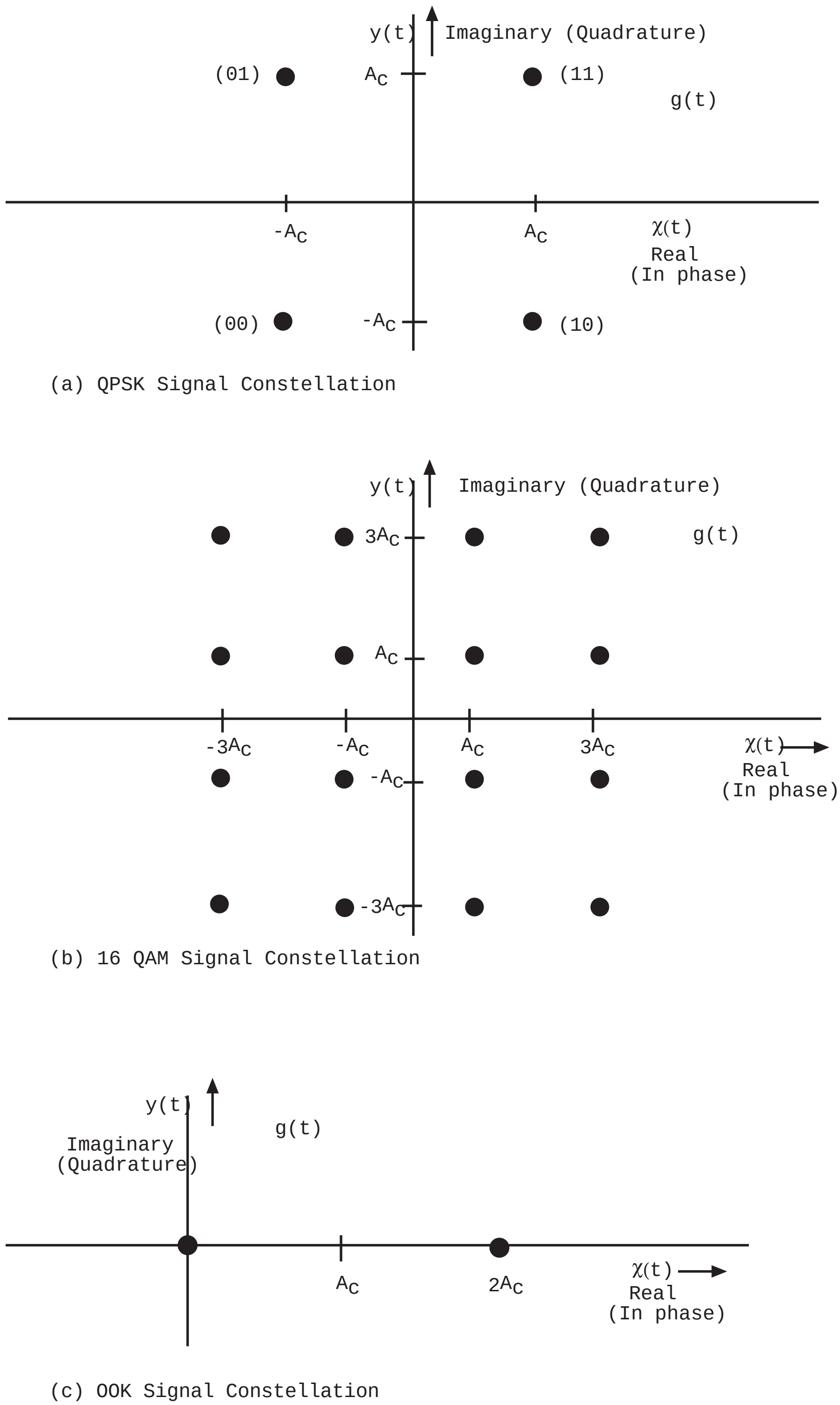

FIGURE 1.5: Signal constellations (permitted values of the complex envelope).

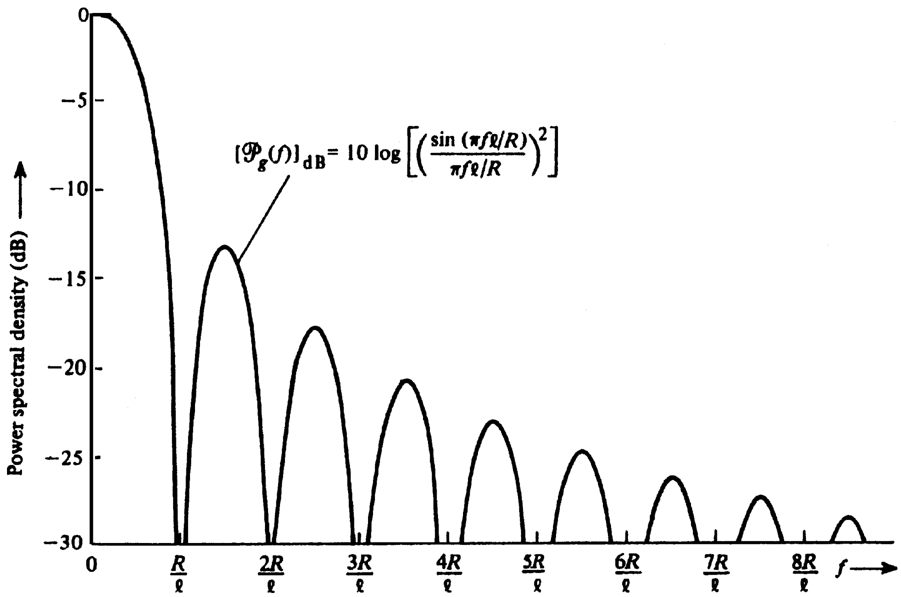

FIGURE 1.6: PSD for the complex envelope of MPSK and QAM where $M = 2^\ell$ and R is bit rate (positive frequencies shown). *Source:* Couch, L.W., II. 1997. *Digital and Analog Communication Systems*, 5th ed., Prentice Hall, Upper Saddle River, NJ, p. 350. With permission.

Baseband waveform: The spectrum of the waveform is nonzero for frequencies near $f = 0$.

Complex envelope: The function $g(t)$ of a bandpass waveform $v(t)$ where the bandpass waveform is described by

$$v(t) = Re\left\{g(t)e^{j\omega_c t}\right\}$$

Fourier transform: If $w(t)$ is a waveform, then the Fourier transform of $w(t)$ is

$$W(f) = \Im[w(t)] = \int_{-\infty}^{\infty} w(t)e^{-j2\pi ft}dt$$

where f has units of hertz.

Modulated signal: The bandpass signal

$$s(t) = Re\left\{g(t)e^{j\omega_c t}\right\}$$

where fluctuations of $g(t)$ are caused by the information source such as audio, video, or data.

Modulation: The information source, $m(t)$, that causes fluctuations in a bandpass signal.

Real envelope: The function $R(t) = |g(t)|$ of a bandpass waveform $v(t)$ where the bandpass waveform is described by

$$v(t) = Re\left\{g(t)e^{j\omega_c t}\right\}$$

Signal constellation: The permitted values of the complex envelope for a digital modulating source.

References

[1] Couch, L.W., II, *Digital and Analog Communication Systems*, 5th ed., Prentice Hall, Upper Saddle River, NJ, 1997.

Further Information

[1] Bedrosian, E., The analytic signal representation of modulated waveforms. *Proc. IRE*, vol. 50, October, 2071–2076, 1962.

[2] Couch, L.W., II, *Modern Communication Systems: Principles and Applications*, Macmillan Publishing, New York, (now Prentice Hall, Upper Saddle River, NJ), 1995.

[3] Dugundji, J., Envelopes and pre-envelopes of real waveforms. *IRE Trans. Information Theory*, vol. IT-4, March, 53–57, 1958.

[4] Voelcker, H.B., Toward the unified theory of modulation—Part I: Phase-envelope relationships. *Proc. IRE*, vol.54, March, 340–353, 1966.

[5] Voelcker, H.B., Toward the unified theory of modulation—Part II: Zero manipulation. *Proc. IRE*, vol. 54, May, 735–755, 1966.

[6] Ziemer, R.E. and Tranter, W.H., *Principles of Communications*, 4th ed., John Wiley and Sons, New York, 1995.